

# SINGULARITY OF THE VARIETIES OF REPRESENTATIONS OF LATTICES IN SOLVABLE LIE GROUPS

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**ABSTRACT.** For a lattice  $\Gamma$  of a simply connected solvable Lie group  $G$ , we describe the analytic germ in the variety of representations of  $\Gamma$  at the trivial representation as an analytic germ which is linearly embedded in the analytic germ associated with the nilpotent Lie algebra determined by  $G$ . By this description, under certain assumption, we study the singularity of the analytic germ in the variety of representations of  $\Gamma$  at the trivial representation by using the Kuranishi space construction. By a similar technique, we also study deformations of holomorphic structures of trivial vector bundles over complex parallelizable solvmanifolds.

## 1. INTRODUCTION

Let  $X$  be an analytic germ in  $\mathbb{C}^n$  at the origin defined by analytic equations

$$f_1(z) = 0, \dots, f_k(z) = 0.$$

We say that  $X$  is cut out by polynomial equations of degree at most  $\nu$  if

$$f_1(z), \dots, f_k(z)$$

are polynomial functions of degree at most  $\nu$  with trivial linear terms. We say that an analytic germ  $Y$  is linearly embedded in  $X$  if for a subspace  $V \subset \mathbb{C}^n$ , the germ  $Y$  is equivalent to an analytic germ in  $V$  at the origin defined by analytic equations

$$f_1(z) = 0, \dots, f_k(z) = 0, \quad z \in V.$$

If  $X$  is cut out by polynomial equations of degree at most  $\nu$  and  $Y$  is linearly embedded in  $X$ , then  $Y$  is also cut out by polynomial equations of degree at most  $\nu$ .

Let  $\Gamma$  be a finitely generated group,  $A$  a linear algebraic group with Lie algebra  $\mathfrak{a}$  and  $R(\Gamma, A)$  the set of homomorphisms  $\Gamma \rightarrow A$ . Then  $R(\Gamma, A)$  can be considered as an affine algebraic variety. For a representation  $\rho \in R(\Gamma, A)$  we are interested in the analytic germ  $(R(\Gamma, A), \rho)$ . The singularity of the analytic germ  $(R(\Gamma, A), \rho)$  can be considered as an obstruction of deformations of  $\rho$ .

If  $\Gamma$  is the fundamental group of a manifold  $M$ , we can geometrically describe the analytic germ  $(R(\Gamma, A), \rho)$  by using the deformation theory of differential graded Lie algebras (for short, DGLAs) developed by Goldman and Millson [6], [7]. By such technique and the Hodge theory of local systems over Kähler manifolds studied by Simpson [14], if  $\Gamma$  is a Kähler group (i.e. a group which can be the fundamental group of a compact Kähler manifold), then for a semisimple representation  $\rho \rightarrow$

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$\mathrm{GL}_m(\mathbb{C})$ , the analytic germ  $(R(\Gamma, \mathrm{GL}_m(\mathbb{C})), \rho)$  is cut out by polynomial equations of degree at most 2.

However in general the analytic germ  $(R(\Gamma, A), \rho)$  is not cut out by polynomial equations of degree at most 2. In [6], Goldman and Millson observed that for a lattice  $\Gamma$  in the three dimensional real Heisenberg group, the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is equivalent to a cubic cone. In this paper we consider a certain class of groups which contains this example.

Let  $\Gamma$  be a lattice in a simply connected solvable Lie group  $G$ . Then the solvmanifold  $G/\Gamma$  is an aspherical manifold with the fundamental group  $\Gamma$ . The purpose of this paper is to study the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$ .

For a manifold  $M$  the analytic germ at the trivial representation of the fundamental group of  $M$  can be studied by the differential graded algebra (for short, DGA)  $A^*(M)$  of differential forms on  $M$ . The following result is known.

**Theorem 1.1** ([6, 7], [4]). *Let  $M$  be a compact manifold with the fundamental group  $\Gamma$ . Suppose that we have a finite dimensional sub-DGA  $C^* \subset A^*(M) \otimes \mathbb{C}$  such that the inclusion induces a cohomology isomorphism and  $C^0 = \mathbb{C}$ . Then the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is equivalent to the analytic germ  $(F(C^*, \mathfrak{a}), 0)$  at the origin 0 for the affine variety*

$$F(C^*, \mathfrak{a}) = \left\{ \omega \in C^* \otimes \mathfrak{a} : d\omega + \frac{1}{2}[\omega, \omega] = 0 \right\}.$$

Consider a solvmanifold  $G/\Gamma$ , Lie algebra  $\mathfrak{g}$  of  $G$  and the cochain complex  $\bigwedge \mathfrak{g}^*$  which is regarded as a differential graded algebra of left- $G$ -invariant forms on  $G/\Gamma$ . Suppose that  $G$  is completely solvable. In [8] Hattori proved that the inclusion  $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$  induces a cohomology isomorphism. By Theorem 1.1 and Hattori's theorem, in [4], Dimca and Papadima remarked that the analytic germ  $(R(\Gamma, A), \mathbf{1})$  is equivalent to the analytic germ  $(F(\bigwedge \mathfrak{g}^*, \mathfrak{a}), 0)$  at the origin 0. However, for a general solvmanifold  $G/\Gamma$ , the inclusion  $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$  does not induce a cohomology isomorphism.

In this paper, we consider general solvmanifolds. Let  $\mathfrak{g}$  be a solvable Lie algebra. Then we can define the nilpotent Lie algebra  $\mathfrak{u}$  called nilshadow of  $\mathfrak{g}$  which is uniquely determined by  $\mathfrak{g}$ , as shown in [5].

**Theorem 1.2** ([9]). *Let  $G$  be a simply connected solvable Lie group with a lattice  $\Gamma$  and  $\mathfrak{g}$  the Lie algebra of  $G$ . We consider the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$ . Then we have a sub-DGA  $A_\Gamma^* \subset A^*(G/\Gamma) \otimes \mathbb{C}$  such that:*

- *The inclusion  $A_\Gamma^* \subset A^*(G/\Gamma) \otimes \mathbb{C}$  induces an isomorphism in cohomology.*
- *$A_\Gamma^*$  can be regarded as a sub-DGA of  $\bigwedge \mathfrak{u}^* \otimes \mathbb{C}$ .*

See Section 2 for the constructions of the nilshadow and DGA  $A_\Gamma^*$ . By this theorem and Theorem 1.1, the analytic germ  $(R(\Gamma, A), \mathbf{1})$  is equivalent to the analytic germ  $(F(A_\Gamma^*, \mathfrak{a}), 0)$ . By the second assertion of the theorem we have the following theorem.

**Theorem 1.3.** *Let  $\Gamma$  be a lattice in a simply connected solvable Lie group  $G$  and  $\mathfrak{g}$  the Lie algebra of  $G$ . Let  $A$  be a linear algebraic group with the Lie algebra  $\mathfrak{a}$ . Consider the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$ . Then the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is linearly embedded in the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{a}), 0)$  at the*

origin 0 for the affine variety

$$F(\bigwedge \mathfrak{u}^*, \mathfrak{a}) = \left\{ \omega \in \bigwedge \mathfrak{u}^* \otimes \mathfrak{a} : d\omega + \frac{1}{2}[\omega, \omega] = 0 \right\}$$

This theorem is useful for estimating the singularity of the analytic germ  $(R(\Gamma, A), \mathbf{1})$ .

Let  $\mathfrak{n}$  be a  $\nu$ -step nilpotent Lie algebra. Consider the lower central series

$$\mathfrak{n} = \mathfrak{n}^{(1)} \supset \mathfrak{n}^{(2)} \supset \cdots \supset \mathfrak{n}^{(\nu)} (\neq \{0\}) \supset \mathfrak{n}^{(\nu+1)} = \{0\}$$

where  $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ . Take a subspace  $\mathfrak{a}^{(i)}$  such that  $\mathfrak{n}^{(i)} = \mathfrak{n}^{(i+1)} \oplus \mathfrak{a}^{(i)}$ . We have

$$\mathfrak{n} = \mathfrak{a}^{(1)} \oplus \mathfrak{a}^{(2)} \oplus \cdots \oplus \mathfrak{a}^{(\nu)}.$$

It is known that  $[\mathfrak{n}^{(i)}, \mathfrak{n}^{(j)}] \subset \mathfrak{n}^{(i+j)}$  (see [2]). A nilpotent Lie algebra  $\mathfrak{n}$  is called naturally graded if we can choose subspaces  $\mathfrak{a}^{(i)}$  such that  $[\mathfrak{a}^{(i)}, \mathfrak{a}^{(j)}] \subset \mathfrak{a}^{(i+j)}$ .

We prove the following proposition by using the construction of Kuranishi spaces of DGLAs.

**Proposition 1.4.** *Let  $\mathfrak{n}$  be a  $\nu$ -step naturally graded nilpotent Lie algebra and  $\mathfrak{g}$  a Lie algebra. Then the analytic germ  $(F(\bigwedge \mathfrak{n}^*, \mathfrak{g}), 0)$  is cut out by polynomial equations of degree at most  $\nu + 1$ .*

By this proposition and Theorem 1.3, we have the following theorem.

**Theorem 1.5.** *Let  $\Gamma$  be a lattice in a simply connected solvable Lie group  $G$  and  $\mathfrak{g}$  the Lie algebra of  $G$ . Let  $A$  be a linear algebraic group with a Lie algebra  $\mathfrak{a}$ . Consider the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$ . We suppose that the Lie algebra  $\mathfrak{u}$  is  $\nu$ -step naturally graded. Then the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is cut out by polynomial equations of degree at most  $\nu + 1$ .*

Let  $\mathfrak{n}$  be a two-step nilpotent Lie algebra. For any complement  $\mathfrak{a}^{(1)}$  of  $\mathfrak{n}^{(2)}$  in  $\mathfrak{n}$ , we have  $\mathfrak{n} = \mathfrak{a}^{(1)} \oplus \mathfrak{n}^{(2)}$  and  $[\mathfrak{a}^{(1)}, \mathfrak{a}^{(1)}] \subset \mathfrak{n}^{(2)}$  and so a two-step nilpotent Lie algebra  $\mathfrak{n}$  is naturally graded. Hence as an application of Theorem 1.5, we have the following Corollary

**Corollary 1.6.** *Let  $\Gamma$  be a lattice in a simply connected solvable Lie group  $G$  and  $\mathfrak{g}$  the Lie algebra of  $G$ . Let  $A$  be a linear algebraic group with a Lie algebra  $\mathfrak{a}$ . Consider the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$ . We suppose that the Lie algebra  $\mathfrak{u}$  is two-step nilpotent. Then the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is cut out by polynomial equations of degree at most 3.*

## 2. NILSHADOWS AND COHOMOLOGY OF SOLVMANIFOLDS

Let  $\mathfrak{g}$  be a solvable  $K$ -Lie algebra for  $K = \mathbb{R}$  or  $\mathbb{C}$ . Let  $\mathfrak{n}$  be the nilradical of  $\mathfrak{g}$ . There exists a subvector space (not necessarily Lie algebra)  $V$  of  $\mathfrak{g}$  so that  $\mathfrak{g} = V \oplus \mathfrak{n}$  as the direct sum of vector spaces and for any  $A, B \in V$   $(\text{ad}_A)_s(B) = 0$  where  $(\text{ad}_A)_s$  is the semi-simple part of  $\text{ad}_A$  (see [5, Proposition III.1.1]). We define the map  $\text{ad}_s : \mathfrak{g} \rightarrow D(\mathfrak{g})$  as  $\text{ad}_{sA+X} = (\text{ad}_A)_s$  for  $A \in V$  and  $X \in \mathfrak{n}$ . Then we have  $[\text{ad}_s(\mathfrak{g}), \text{ad}_s(\mathfrak{g})] = 0$  and  $\text{ad}_s$  is linear (see [5, Proposition III.1.1]). Since we have  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$ , the map  $\text{ad}_s : \mathfrak{g} \rightarrow D(\mathfrak{g})$  is a representation and the image  $\text{ad}_s(\mathfrak{g})$  is abelian and consists of semi-simple elements. Let  $\bar{\mathfrak{g}} = \text{Im } \text{ad}_s \ltimes \mathfrak{g}$  and

$$\mathfrak{u} = \{X - \text{ad}_{sX} \in \bar{\mathfrak{g}} \mid X \in \mathfrak{g}\}.$$

Then we have  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{u}$  and  $\mathfrak{u}$  is the nilradical of  $\bar{\mathfrak{g}}$  (see [5]). Hence we have  $\bar{\mathfrak{g}} = \text{Im } \text{ad}_s \ltimes \mathfrak{u}$ . It is known that the structure of the Lie algebra  $\mathfrak{u}$  is independent of a choice of a subvector space  $V$  (see [5, Corollary III.3.6]).

**Lemma 2.1.** ([9, Lemma 2.2]) *Suppose  $\mathfrak{g} = \mathbb{R}^k \ltimes_{\phi} \mathfrak{n}$  such that  $\phi$  is a semi-simple action and  $\mathfrak{n}$  is nilpotent. Then the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$  is the direct sum  $\mathbb{R}^k \oplus \mathfrak{n}$ .*

Let  $G$  be a simply connected solvable Lie group with the  $\mathbb{R}$ -Lie algebra  $\mathfrak{g}$ . We denote by  $\text{Ad}_s : G \rightarrow \text{Aut}(\mathfrak{g})$  the extension of  $\text{ad}_s$ . Then  $\text{Ad}_s(G)$  is diagonalizable. Let  $X_1, \dots, X_n$  be a basis of  $\mathfrak{g} \otimes \mathbb{C}$  such that  $\text{Ad}_s$  is represented by diagonal matrices. Then we have  $\text{Ad}_{sg} X_i = \alpha_i(g) X_i$  for characters  $\alpha_i$  of  $G$ . Let  $x_1, \dots, x_n$  be the dual basis of  $X_1, \dots, X_n$ .

We suppose  $G$  has a lattice  $\Gamma$ . Then we consider the sub-DGA  $A_{\Gamma}^*$  of the de Rham complex  $A^*(G/\Gamma) \otimes \mathbb{C}$  which is given by

$$A_{\Gamma}^p = \left\langle \alpha_I x_I \mid \begin{array}{l} I \subset \{1, \dots, n\}, \\ (\alpha_I)|_{\Gamma} = 1 \end{array} \right\rangle.$$

where for a multi-index  $I = \{i_1, \dots, i_p\}$  we write  $x_I = x_{i_1} \wedge \dots \wedge x_{i_p}$ , and  $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$ .

**Theorem 2.2.** ([9, Corollary 7.6]) *Let  $G$  be a simply connected solvable Lie group with a lattice  $\Gamma$ . Then we have :*

- *The inclusion  $A_{\Gamma}^* \subset A^*(G/\Gamma) \otimes \mathbb{C}$  induces an isomorphism in cohomology.*
- *$A_{\Gamma}^*$  can be regarded as a sub-DGA of  $\bigwedge \mathfrak{u}^* \otimes \mathbb{C}$ .*

We explain the second assertion more precisely. We consider the subspace  $\tilde{\mathfrak{u}} = \langle \alpha_1^{-1} X_1, \dots, \alpha_n^{-1} X_n \rangle$  of the space of complex valued vector fields on  $G$ . Then  $\tilde{\mathfrak{u}} = \langle \alpha_1^{-1} X_1, \dots, \alpha_n^{-1} X_n \rangle$  is a Lie sub-algebra of the Lie algebra of vector fields and the map

$$\tilde{\mathfrak{u}} \ni \alpha_i^{-1} X_i \mapsto X_i - \text{ad}_{sX_i} \in \mathfrak{u} \otimes \mathbb{C}$$

is a Lie algebra isomorphism where  $\mathfrak{u}$  is the nilshadow of  $\mathfrak{g}$  (see [9, Proof of Lemma 5.2]).

*Example 1.* Let  $\mathfrak{g}$  be a 4-dimensional Lie algebra such that

- $\mathfrak{g} = \langle T, X, Y, Z \rangle$
- $[T, X] = X, [T, Y] = -Y, [X, Y] = Z$ .

Then we have the splitting  $\mathfrak{g} = \langle T \rangle \ltimes \langle X, Y, Z \rangle$  such that  $\langle X, Y, Z \rangle$  is the three dimensional real Heisenberg Lie algebra  $\mathfrak{h}(3)$  and the action of  $\langle T \rangle$  is semi-simple. Hence by Lemma 2.1, the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$  is given by  $\mathfrak{u} = \mathbb{R} \oplus \mathfrak{h}(3)$ . Hence as similar to [6, Example 9.1], the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{a}), 0)$  is equivalent to a cubic cone.

Consider the simply connected solvable Lie group  $G$  whose Lie algebra is  $\mathfrak{g}$ . Then  $G$  has a lattice  $\Gamma$  [13]. We can easily show that the DGA  $A^*(G/\Gamma)$  is formal and hence the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is cut out by polynomial equations of degree at most 2. Hence  $(R(\Gamma, A), \mathbf{1})$  is linearly embedded in the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{a}), 0)$  but its singularity is different from  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{a}), 0)$ .

By Lemma 2.1, we give one more corollary of Theorem 1.5.

**Corollary 2.3.** *Let  $\mathfrak{g} = \mathbb{R}^k \ltimes_{\phi} \mathfrak{n}$  such that  $\phi$  is a semi-simple action and  $\mathfrak{n}$  is a  $\nu$ -step naturally graded nilpotent Lie algebra. Consider the simply connected solvable Lie group  $G$  whose Lie algebra is  $\mathfrak{g}$ . Suppose  $G$  has a lattice  $\Gamma$ . Then the analytic germ  $(R(\Gamma, A), \mathbf{1})$  at the trivial representation  $\mathbf{1}$  is cut out by polynomial equations of degree at most  $\nu + 1$ .*

## 3. PROOF OF PROPOSITION 1.4

**3.1. Finite-dimensional DGAs of Poincaré duality type.** Let  $A^*$  be a finite-dimensional graded commutative  $\mathbb{C}$ -algebra.

**Definition 3.1** ([10]).  $A^*$  is of Poincaré duality type (PD-type) if the following conditions hold:

- $A^{*<0} = 0$  and  $A^0 = \mathbb{C}1$  where 1 is the identity element of  $A^*$ .
- For some positive integer  $n$ ,  $A^{*>n} = 0$  and  $A^n = \mathbb{C}v$  for  $v \neq 0$ .
- For any  $0 < i < n$  the bi-linear map  $A^i \times A^{n-i} \ni (\alpha, \beta) \mapsto \alpha \cdot \beta \in A^n$  is non-degenerate.

Suppose  $A^*$  is of PD-type. Let  $h$  be a Hermitian metric on  $A^*$  which is compatible with the grading. Take  $v \in A^n$  such that  $h(v, v) = 1$ . Define the  $\mathbb{C}$ -anti-linear map  $\bar{*} : A^i \rightarrow A^{n-i}$  as  $\alpha \cdot \bar{*}\beta = h(\alpha, \beta)v$ .

**Definition 3.2** ([10]). A finite-dimensional DGA  $(A^*, d)$  is of PD-type if the following conditions hold:

- $A^*$  is a finite-dimensional graded  $\mathbb{C}$ -algebra of PD-type.
- $dA^{n-1} = 0$  and  $dA^0 = 0$ .

Let  $(A^*, d)$  be a finite-dimensional DGA of PD-type. Denote  $d^* = -\bar{*}d\bar{*}$ .

**Lemma 3.3** ([10]). We have  $h(d\alpha, \beta) = h(\alpha, d^*\beta)$  for  $\alpha \in A^{i-1}$  and  $\beta \in A^i$ .

Define  $\Delta = dd^* + d^*d$ . and  $\mathcal{H}^*(A) = \ker \Delta$ . By Lemma 3.3 and finiteness of the dimension of  $A^*$ , we can easily show the following lemma.

**Lemma 3.4** ([10]). We have the Hodge decomposition

$$A^r = \mathcal{H}^r(A) \oplus \Delta(A^r) = \mathcal{H}^r(A) \oplus d(A^{r-1}) \oplus d^*(A^{r+1}).$$

By this decomposition, the inclusion  $\mathcal{H}^*(A) \subset A^*$  induces a isomorphism

$$\mathcal{H}^p(A) \cong H^p(A)$$

of vector spaces.

We denote by  $H$  the projection  $H : A^p \rightarrow \mathcal{H}^p(A)$  and define the operator  $G$  as the composition  $\Delta_{|\Delta(A^p)}^{-1} \circ (\text{id} - H)$ . Let  $\beta : A^* \rightarrow dA^{*-1}$  be the projection for the decomposition

$$A^r = \mathcal{H}^r(A) \oplus d(A^{r-1}) \oplus d^*(A^{r+1}).$$

The restriction map  $d : d^*(A^*) \rightarrow d(A^{*-1})$  is an isomorphism. Take the inverse  $d^{-1} : d(A^{*-1}) \rightarrow d^*(A^*)$ . Consider the map  $d^*G : A^* \rightarrow A^{*-1}$ . Then for  $\omega \in \mathcal{H}^r(A)$ ,  $d^*x \in d^*(A^r)$  and  $d^*y \in d^*(A^{r+1})$ , we have

$$d^*G(\omega + dd^*x + d^*y) = d^*(dd^*)^{-1}dd^*x = d^*x.$$

Hence we have  $d^*G = d^{-1} \circ \beta$ .

**3.2. Kuranishi spaces of finite-dimensional DGLAs.** Let  $L^*$  be a finite-dimensional DGLA with a differential  $d$ . Consider the splitting  $d(L^p) \rightarrow L^p$  for the short exact sequence

$$0 \longrightarrow \ker d|_{L^p} \longrightarrow L^p \xrightarrow{d} d(L^p) \longrightarrow 0$$

and the splitting  $H^p(L^*) \rightarrow \ker d|_{L^p}$  for the short exact sequence

$$0 \longrightarrow d(L^{p-1}) \longrightarrow \ker d|_{L^p} \longrightarrow H^p(L^*) \longrightarrow 0.$$

Denote by  $\mathcal{A}^p$  and  $\mathcal{H}^p$  the images of the splittings  $d(L^p) \rightarrow L^p$  and  $H^p(L^*) \rightarrow \ker d|_{L^p}$  respectively. Then we have

$$L^p = \mathcal{H}^p \oplus d(L^{p-1}) \oplus \mathcal{A}^p.$$

Consider the projections  $\beta^* : L^* \rightarrow d(L^{*-1})$ ,  $H : L^* \rightarrow \mathcal{H}^*$  and  $\alpha^\beta : L^* \rightarrow \mathcal{A}^*$ . Since the restriction  $d : \mathcal{A}^p \rightarrow d(L^p)$  is an isomorphism, we have the inverse  $d^{-1} : d(L^p) \rightarrow \mathcal{A}^p$  of  $d : \mathcal{A}^p \rightarrow d(L^p)$ . We define  $\delta = d^{-1} \circ \beta : L^{p+1} \rightarrow L^p$ . Define the map  $F : L^1 \rightarrow L^1$  as

$$F(\zeta) = \zeta + \frac{1}{2}\delta[\zeta, \zeta].$$

Then by the inverse function theorem, on a small ball  $B$  in  $L^1$ , the map  $F$  is an analytic diffeomorphism. Then the Kuranishi space  $\mathcal{K}(L^*)$  is defined by

$$\mathcal{K}(L^*) = \{\eta \in F(B) \cap \mathcal{H}^1 : H([F^{-1}(\eta), F^{-1}(\eta)]) = 0\}.$$

It is known that the analytic germ  $(\mathcal{K}(L^*), 0)$  is equivalent to the germ at the origin for the variety

$$\left\{ \zeta \in L^1 : d\zeta + \frac{1}{2}[\zeta, \zeta] = 0, \delta\zeta = 0 \right\}$$

(see [7, Theorem 2.6]). In particular, if  $d(L^0) = 0$ , then  $\mathcal{K}(L^*)$  is equivalent to the germ at the origin for the variety

$$\left\{ \zeta \in L^1 : d\zeta + \frac{1}{2}[\zeta, \zeta] = 0 \right\}.$$

Take a basis  $\zeta_1, \dots, \zeta_m$  of  $\mathcal{H}^1$ . For parameters  $t = (t_i)$ , we consider the formal power series  $\phi(t) = \sum_r \phi_r(t)$  with values in  $L^1$  given inductively by  $\phi_1(t) = \sum t_i \zeta_i$  and

$$\phi_r(t) = -\frac{1}{2} \sum_{s=1}^{r-1} \delta[\phi_s(t), \phi_{r-s}(t)].$$

Then  $F^{-1}$  is given by  $\phi_1(t) \mapsto \phi(t)$  and the Kuranishi space  $\mathcal{K}(L^*)$  is an analytic germ in  $\mathbb{C}^m$  at the origin defined by equations

$$H([\phi(t), \phi(t)]) = 0.$$

Let  $A$  be a finite-dimensional DGA of PD-type and  $\mathfrak{g}$  a Lie algebra, and consider the DGLA  $A^* \otimes \mathfrak{g}$ . Then we have the Hodge decomposition

$$A^* \otimes \mathfrak{g} = \mathcal{H}^p(A) \otimes \mathfrak{g} \oplus d(A^{p-1}) \otimes \mathfrak{g} \oplus d^*(A^{p+1}) \otimes \mathfrak{g}$$

as above with  $\delta = d^*G \otimes \text{id}$ . Take a basis  $\zeta_1, \dots, \zeta_m$  of  $\mathcal{H}^1(A^*) \otimes \mathfrak{g}$ . For parameters  $t = (t_i)$ , we consider the formal power series  $\phi(t) = \sum_r \phi_r(t)$  with values in  $A^1 \otimes \mathfrak{g}$  given inductively by  $\phi_1(t) = \sum t_i \zeta_i$  and

$$\phi_r(t) = -\frac{1}{2} \sum_{s=1}^{r-1} d^*G \otimes \text{id}[\phi_s(t), \phi_{r-s}(t)].$$

By the above argument we have the following lemma.

**Lemma 3.5.** *The analytic germ  $(F(A^*, \mathfrak{g}), 0)$  is equivalent to the analytic germ in  $\mathbb{C}^m$  at the origin defined by equations*

$$H([\phi(t), \phi(t)]) = 0.$$

**3.3. Nilpotent Lie algebras.** Let  $\mathfrak{n}$  be a  $\nu$ -step nilpotent  $K$ -Lie algebra for  $K = \mathbb{R}$  or  $\mathbb{C}$ . Consider the lower central series

$$\mathfrak{n} = \mathfrak{n}^{(1)} \supset \mathfrak{n}^{(2)} \supset \cdots \supset \mathfrak{n}^{(\nu)} \supset \mathfrak{n}^{(\nu+1)} = \{0\}$$

where  $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ . Take a subspace  $\mathfrak{a}^{(i)}$  such that  $\mathfrak{n}^{(i)} = \mathfrak{n}^{(i+1)} \oplus \mathfrak{a}^{(i)}$ . We have

$$\mathfrak{n} = \mathfrak{a}^{(1)} \oplus \mathfrak{a}^{(2)} \oplus \cdots \oplus \mathfrak{a}^{(\nu)}.$$

Consider the dual spaces  $\mathfrak{n}^*$  and  $\mathfrak{a}^{(i)*}$  of  $\mathfrak{n}$  and  $\mathfrak{a}^{(i)}$  respectively. We consider the cochain complex  $\bigwedge \mathfrak{n}^*$  of the Lie algebra with the differential  $d$ . Then  $\bigwedge \mathfrak{n}^*$  is a finite-dimensional DGA of PD-type. We have

$$\bigwedge \mathfrak{n}^* = \left( \bigwedge \mathfrak{a}^{(1)*} \right) \wedge \cdots \wedge \left( \bigwedge \mathfrak{a}^{(\nu)*} \right).$$

We have

$$H^1(\mathfrak{n}) = \ker d_{\bigwedge^1 \mathfrak{n}^*} = \mathfrak{a}^{(1)*}.$$

**Lemma 3.6.**

$$\ker d_{\bigwedge^2 \mathfrak{n}^*} \subset \bigoplus_{i+j \leq \nu+1, i \leq j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}.$$

*Proof.* Let  $\sigma \in \ker d_{\bigwedge^2 \mathfrak{n}^*}$ . For a positive integer  $k < \nu$ , we say that  $\sigma$  is  $k$ -decomposable if we have a decomposition

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3$$

such that:

$$\begin{aligned} \bullet & \quad \sigma_1 \in \bigoplus_{i+j \leq \nu+1, i \leq j, k < j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}. \\ \bullet & \quad \sigma_2 \in \bigoplus_{i \leq k} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(k)*}. \\ \bullet & \quad \sigma_3 \in \bigoplus_{i \leq j, j < k} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}. \end{aligned}$$

If  $k \leq \frac{\nu+1}{2}$ , then we have

$$\sigma \in \bigoplus_{i+j \leq \nu+1, i \leq j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}.$$

Consider the case  $\frac{\nu+1}{2} < k$ . For  $X, Y \in \mathfrak{n}$  and  $Z \in \mathfrak{n}^{(k)}$ , we have  $\sigma_1([X, Y], Z) = 0$ ,  $\sigma_2(X, [Y, Z]) = 0$ ,  $\sigma_2(Y, [X, Z]) = 0$ ,  $\sigma_3([X, Y], Z) = 0$ ,  $\sigma_3(X, [Y, Z]) = 0$  and  $\sigma_3(Y, [X, Z]) = 0$ . By  $d\sigma = 0$ , we have

$$\sigma_2([X, Y], Z) = \sigma_1(X, [Y, Z]) - \sigma_1(Y, [X, Z]).$$

Taking  $X \in \mathfrak{n}$  and  $Y \in \mathfrak{n}^{(l-1)}$  such that  $\nu+1 < k+l$ , we have

$$\sigma_2([X, Y], Z) = 0.$$

Hence for  $W \in \mathfrak{n}^{(l)}$  and  $Z \in \mathfrak{n}^{(k)}$  such that  $\nu+1 < k+l$ , we have

$$\sigma_2(W, Z) = 0.$$

Thus we have

$$\sigma_2 \in \bigoplus_{i+k \leq \nu+1, i \leq k} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(k)*}.$$

Hence taking  $\sigma'_1 = \sigma_1 + \sigma_2$  and  $\sigma_3 = \sigma'_2 + \sigma'_3$  such that

$$\sigma'_2 \in \bigoplus_{i \leq k-1} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(k-1)*}$$

and

$$\sigma'_3 \in \bigoplus_{i \leq j, j < k-1} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*},$$

by the decomposition  $\sigma = \sigma'_1 + \sigma'_2 + \sigma'_3$ ,  $\sigma$  is  $(k-1)$ -decomposable. Thus we can say that if  $\sigma$  is  $k$ -decomposable and  $\frac{\nu+1}{2} < k-l-1$  for an integer  $l$ , then  $\sigma$  is also  $(k-l)$ -decomposable. Take  $l$  such that  $k-l \leq \frac{\nu+1}{2}$ . Then we can say

$$\sigma \in \bigoplus_{i+j \leq \nu+1, i \leq j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}.$$

Hence it is sufficient to show the above decomposition of  $\sigma$  for  $k = \nu - 1$ . This was shown in [1, Lemma 2.8]. Hence the Lemma follows.  $\square$

It is known that  $[\mathfrak{n}^{(i)}, \mathfrak{n}^{(j)}] \subset \mathfrak{n}^{(i+j)}$  (see [2]) and hence we have

$$d\left(\mathfrak{a}^{(k)*}\right) \subset \bigoplus_{i+j \leq k, i \leq j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}.$$

**Definition 3.7.** A nilpotent Lie algebra  $\mathfrak{n}$  is called naturally graded if we can take subspaces  $\mathfrak{a}^{(i)} \subset \mathfrak{n}$  such that  $\mathfrak{n}^{(i)} = \mathfrak{n}^{(i+1)} \oplus \mathfrak{a}^{(i)}$  and  $[\mathfrak{a}^{(i)}, \mathfrak{a}^{(j)}] \subset \mathfrak{a}^{(i+j)}$  for each  $i, j$  where

$$\mathfrak{n} = \mathfrak{n}^{(1)} \supset \mathfrak{n}^{(2)} \supset \dots \supset \mathfrak{n}^{(\nu)} \supset \mathfrak{n}^{(\nu+1)} = \{0\}$$

is the lower central series of  $\mathfrak{n}$ .

If  $\mathfrak{n}$  is naturally graded, then we have

$$d\left(\mathfrak{a}^{(k)*}\right) \subset W_k$$

where  $W_k = \bigoplus_{i+j=k, i \leq j} \mathfrak{a}^{(i)*} \wedge \mathfrak{a}^{(j)*}$ .

Let  $g$  be a Hermitian metric on  $\mathfrak{n}$  such that the sum

$$\mathfrak{n} = \mathfrak{a}^{(1)} \oplus \mathfrak{a}^{(2)} \oplus \dots \oplus \mathfrak{a}^{(\nu)}$$

is an orthogonal direct sum. Then  $g$  give a Hermitian metric on the finite-dimensional DGA  $\bigwedge \mathfrak{n}^*$  of PD-type. Consider the decomposition

$$\bigwedge^r \mathfrak{n}^* = \mathcal{H}^r(\bigwedge \mathfrak{n}^*) \oplus d\left(\bigwedge^{r-1} \mathfrak{n}^*\right) \oplus d^*\left(\bigwedge^{r+1} \mathfrak{n}^*\right).$$

Then

$$\bigwedge^2 \mathfrak{n}^* = W_1 \oplus W_2 \oplus \dots \oplus W_{2\nu}$$

is an orthogonal direct sum and we have  $d^{-1} \circ \beta(W_k) \subset \mathfrak{a}^{(k)*}$  by  $d(\mathfrak{a}^{(k)*}) \subset W_k$ .

**Proposition 3.8.** *Let  $\mathfrak{n}$  be a  $\nu$ -step naturally graded nilpotent Lie algebra and  $\mathfrak{g}$  a Lie algebra. Then the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{g}), 0)$  is cut out by polynomial equations of degree at most  $\nu + 1$ .*



*Proof.* Take a basis  $\zeta_1, \dots, \zeta_m$  of  $\mathcal{H}^1(\bigwedge \mathfrak{u}^*) \otimes \mathfrak{g}$ . For parameters  $t = (t_i)$ , we consider the formal power series  $\phi(t) = \sum_r \phi_r(t)$  with values in  $L^1$  given inductively by  $\phi_1(t) = \sum t_i \zeta_i$  and

$$\phi_r(t) = -\frac{1}{2} \sum_{s=1}^{r-1} \delta[\phi_s(t), \phi_{r-s}(t)].$$

By Lemma 3.5, the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{g}), 0)$  is equivalent to the analytic germ in  $\mathbb{C}^m$  at the origine defined by equations

$$H([\phi(t), \phi(t)]) = 0$$

where  $H : \bigwedge^* \mathfrak{n}^* \otimes \mathfrak{g} \rightarrow \mathcal{H}^*(\bigwedge \mathfrak{u}^*) \otimes \mathfrak{g}$  is the projection.

We have

$$[\mathfrak{a}^{(i)*} \otimes \mathfrak{g}, \mathfrak{a}^{(j)*} \otimes \mathfrak{g}] \subset W_{i+j} \otimes \mathfrak{g}.$$

By  $d^*G(W_k) = d^{-1} \circ \beta(W_k) \subset \mathfrak{a}^{(k)*}$ , we have

$$d^*G \otimes \text{id}([\mathfrak{a}^{(i)*} \otimes \mathfrak{g}, \mathfrak{a}^{(j)*} \otimes \mathfrak{g}]) \subset \mathfrak{a}^{(i+j)*} \otimes \mathfrak{g}.$$

This implies  $\phi_r(t) \in \mathfrak{a}^{(r)*} \otimes \mathfrak{g}$  and we have

$$\phi(t) = \phi_1(t) + \dots + \phi_\nu(t).$$

By Lemma 3.6, we have  $\mathcal{H}^2(\bigwedge \mathfrak{n}^*) \subset \text{Ker } d_{\bigwedge^2 \mathfrak{n}^*} \subset \bigoplus_{l \leq \nu+1} W_l$  and hence

$$H\left(\bigwedge^2 \mathfrak{n}^* \otimes \mathfrak{g}\right) \subset \text{Ker } d_{\bigwedge^2 \mathfrak{n}^*} \otimes \mathfrak{g} \subset \bigoplus_{l \leq \nu+1} W_l \otimes \mathfrak{g}.$$

Since we have  $[\phi_i(t), \phi_j(t)] \in W_{i+j} \otimes \mathfrak{g}$  by  $\phi_r(t) \in \mathfrak{a}^{(r)*} \otimes \mathfrak{g}$ , we have  $H[\phi_i(t), \phi_j(t)] = 0$  for  $\nu+1 < i+j$ . Hence  $H[\phi(t), \phi(t)] = 0$  are polynomial equations of degree at most  $\nu+1$ . □

#### 4. COMPLEX ANALOGY

**4.1. Complex parallelizable solvmanifolds.** Let  $G$  be a simply connected  $n$ -dimensional complex solvable Lie group. Consider the Lie algebra  $\mathfrak{g}_{1,0}$  (resp.  $\mathfrak{g}_{0,1}$ ) of the left-invariant holomorphic (resp. anti-holomorphic) vector fields on  $G$ . Let  $N$  be the nilradical of  $G$ . We can take a simply connected complex nilpotent subgroup  $C \subset G$  such that  $G = C \cdot N$  (see [3]). Since  $C$  is nilpotent, the map

$$C \ni c \mapsto (\text{Ad}_c)_s \in \text{Aut}(\mathfrak{g}_{1,0})$$

is a homomorphism where  $(\text{Ad}_c)_s$  is the semi-simple part of  $\text{Ad}_s$ .

We have a basis  $X_1, \dots, X_n$  of  $\mathfrak{g}_{1,0}$  such that

$$(\text{Ad}_c)_s = \text{diag}(\alpha_1(c), \dots, \alpha_n(c))$$

for  $c \in C$ . Let  $x_1, \dots, x_n$  be the basis of  $\mathfrak{g}_{1,0}^*$  which is dual to  $X_1, \dots, X_n$ .

**Theorem 4.1.** ([11, Corollary 6.2 and Remark 5]) *Suppose  $G$  has a lattice  $\Gamma$ . Let  $B_\Gamma^*$  be the subcomplex of  $(A^{0,*}(G/\Gamma), \bar{\partial})$  defined as*

$$B_\Gamma^* = \left\langle \frac{\bar{\alpha}_I}{\alpha_I} \bar{x}_I \mid \left( \frac{\bar{\alpha}_I}{\alpha_I} \right)_{|_\Gamma} = 1 \right\rangle$$

where for a multi-index  $I = \{i_1, \dots, i_p\}$  we write  $x_I = x_{i_1} \wedge \dots \wedge x_{i_p}$ , and  $\alpha_I = \alpha_{i_1} \dots \alpha_{i_p}$ . Consider the nilshadow  $\mathfrak{u}$  of the  $\mathbb{C}$ -Lie algebra  $\mathfrak{g}$ . Then we have :

- The inclusion  $B_\Gamma^* \subset A^{0,*}(G/\Gamma)$  induces an isomorphism in cohomology.
- $B_\Gamma^*$  can be regarded as a sub-DGA of  $\bigwedge \mathfrak{u}^*$ .

It is known that a simply connected solvable Lie group  $G$  admitting a lattice  $\Gamma$  is unimodular. Hence we have  $\alpha_1 \cdots \alpha_n = 1$ . For a multi-index  $I \subset \{1, \dots, n\}$  and its complement  $I' = \{1, \dots, n\} - I$ , if  $\left(\frac{\bar{\alpha}_I}{\alpha_I}\right)|_\Gamma = 1$  then  $\left(\frac{\bar{\alpha}_{I'}}{\alpha_{I'}}\right)|_\Gamma = 1$ . Thus the DGA  $B_\Gamma^*$  as in Theorem 4.1 is of PD-type.

**4.2. Deformations of holomorphic vector bundles.** For a compact complex manifold  $(M, J)$  and a holomorphic vector bundle  $E$  over  $M$ , consider

$$L^* = A^{0,*}(M, \text{End}(E))$$

the differential graded Lie algebra of differential forms of  $(0, *)$ -type with values in the holomorphic vector bundle  $\text{End}(E)$  with the Dolbeault operator induced by the holomorphic structure on  $E$ . Then the Kuranishi space  $\mathcal{K}(L^*)$  represents the deformation functor for deformations of holomorphic structures on  $E$  (see [6] and [7]).

Let  $G$  be a simply connected  $n$ -dimensional complex solvable Lie group with a lattice  $\Gamma$ . For the Lie algebra  $\mathfrak{gl}_n(\mathbb{C})$  of complex valued  $n \times n$  matrices, we consider the DGLA  $L^* = A^{0,*}(G/\Gamma) \otimes \mathfrak{gl}_n(\mathbb{C})$ . Then the Kuranishi space  $\mathcal{K}(L^*)$  represents the deformation functor for deformations of holomorphic structures on  $G/\Gamma \times \mathbb{C}^n$  near the trivial holomorphic structure. As an analytic germ, the Kuranishi space  $\mathcal{K}(L^*)$  is an invariant under quasi-isomorphisms between analytic DGLAs. Hence by Theorem 4.1, considering the DGLA  $\overline{L}^* = B_\Gamma^* \otimes \mathfrak{gl}_n(\mathbb{C})$ , the analytic germ  $\mathcal{K}(L^*)$  is equivalent to  $\mathcal{K}(\overline{L}^*)$ . As Section 3.2, the analytic germ  $\mathcal{K}(\overline{L}^*)$  is equivalent to the analytic germ  $(F(B_\Gamma^*, \mathfrak{gl}_n(\mathbb{C})), 0)$ . Hence we have the following theorem.

**Theorem 4.2.** *Let  $G$  be a simply connected complex solvable Lie group with a lattice  $\Gamma$  and  $\mathfrak{g}$  the  $\mathbb{C}$ -Lie algebra of  $G$ . We consider the nilshadow  $\mathfrak{u}$  of  $\mathfrak{g}$ . Then the analytic germ which represents the deformation functor for deformations of holomorphic structures on  $G/\Gamma \times \mathbb{C}^n$  near the trivial holomorphic structure is linearly embedded in the analytic germ  $(F(\bigwedge \mathfrak{u}^*, \mathfrak{gl}_n(\mathbb{C})), 0)$ .*

*Moreover, suppose that the Lie algebra  $\mathfrak{u}$  is  $\nu$ -step naturally graded. Then such analytic germ is cut out by polynomial equations of degree at most  $\nu + 1$ .*

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